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CLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY MODIFIED CATA'S OPERATOR

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ABSTRACT

In this paper, we introduce a new class of harmonic univalent functions defined by modified Cata; s operator. Coefficient estimates, extreme points, distortion bounds and convex combination for functions belonging to this class are obtained and also for a class preserving integral operator.2000 Mathematics Subject Classification: 30C45.

KEYWORDS: *Harmonic Univalent Functions, Cata; S Operator, Extreme Points.*

1. INTRODUCTION

A continuous complex-valued function f = u+iv is defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we can write

$$f = h + \bar{g} \quad (1.1)$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and su¢ cient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)| >$ in D (see [5]).

Denote by S_H the class of functions f of the form (1.1) that are harmonic univalent and sense-preserving in the unit disk

$$U = \{z : |z| < 1\}$$

for which

$$f(0) = f_z(0)\ 1 = 0.$$

Then for $f = h + \bar{g}\ S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \ ;\ g(z) = \sum_{n=1}^{\infty} b_n z^n ,\ |b_1| < 1 \quad (1.2)$$

In [5] Clunie and Shell-Small investigated the class SH as well as its geometric classes and obtained some coefficicient bounds. Since then, there have been several related papers on $\overline{S_H}$ and its classes.

Let SH denote the pclasses of SH consisting of functions $f = h + \bar{g}$ such that h and g given by

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \ ;\ g(z) = \sum_{n=1}^{\infty} (1)^m b_n z^n ,\ |b_1| < 1: \quad (1.3)$$

For $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$

$\mathbb{N}$ = {1, 2,.....},μ ≥ 0 and l ≥ 0; the extended multiplier transformation I m (μ;l) is defined by the following infinite series (see [2]):

$I1(\mu,\alpha)f(z) = z + \sum_{n=2}^{\infty}[\frac{\alpha+\mu(n-1)}{\alpha}]^1 a_n z^n$ (1.4)

It follows from (1.4) that (see [2])

$\mu z(I^1(\mu,l))' = \alpha I2(;l)f(z)\ \alpha I1(\mu,l)f(z)-(1-\mu+l)I^1(\mu,l)f(z)$ (($\mu > 0$) and

$I^{m_1}(\mu,l)(I^{m_2}(\mu,l)f(z)) = I^{m_1+m_2}(\mu,l)f(z) = I^{m_2}(\mu,l)(I^{m_1}(\mu,l)f(z))$ for all integers m1 and m2:

We note that: I0 $(\mu,l)f(z) = f(z)$ and I1 (1; 0)f(z) = $zf^1(z)$:

Also, we can write

I1 $(\mu,l)f(z) = (\Phi^m_{\mu,l} * f)(z)$;

where

$(\Phi^m_{\mu,l} * f)(z) = \sum_{n=2}^{\infty}[\frac{\alpha+\mu(n-1)}{\alpha}]^1 z^n$:

Now we can deÖne the modified Cata's operator as follows:

I (m, μ, l) f(z) = I1 (μ, l) h(z) + (-1)m $\overline{I^1(\mu,l)g(z)}$ (1.5)

where

$$I1(\mu,\alpha)h(z) = z + \sum_{n=2}^{\infty}[\frac{\alpha+\mu(n-1)}{\alpha}]^1 a_n z^n$$

and

$$I1(\mu,\alpha)h(z) = (-1)m + \sum_{n=2}^{\infty}[\frac{\alpha+\mu(n-1)}{\alpha}]^1 a_n z^n$$

For $1 < \gamma \leq 2$ and for all z ∈U; let $SI^1(\mu,l)$ denote the family of harmonic functions f(z) = h + g; where h and g given by (1.2) and satisfying the analytic criterion

$$\text{Re}\ \{\frac{I^m(\mu,l)h(z)+\overline{(-1)^m(\mu,l)g(z)}}{z}\} < \gamma: (1.6)$$

Let $\overline{SI^m(\mu,l)}$ be the subclass of consisting of functions f = h + g such that h and g given by (1.3).

We note that for suitable choices of m; μ, and l; we obtain the following subclasses:

(1) Putting μ = 1 and l = 0; in (1.6), the class

reduces to the class $\overline{SI^m}(\gamma)$ ={ f$\in S_H \text{Re}\frac{D^m h(z)+\overline{(-1)^m(D)^m g(z)}}{z}\}$ < γ; 1< γ ≤2; m ∈ $\mathbb{N}_0$0 z ∈ U}; where Dm is the modified Salagean operator (see [7]), the differential opertor Dm was introduced by Salagean (see [8]);

(2) Putting = 1 and l = 1; in (1.6), the class $\overline{SI^m(1,1,\gamma)}$ reduces to the class reduces to the class $\overline{SI^m}(\gamma)$ ={ f∈ $S_H \text{Re}\frac{I^m h(z)+\overline{(-1)^m(I)^m g(z)}}{z}\}$ < γ;; 1< γ ≤2; m ∈ $Z = \{0, \pm1 \pm 2 \pm 3 \pm \cdots$ ... ... z ∈ U};

where I^m is the modified Uralegaddi-Somanatha operator (see [9]), deÖned as follows:

I mf(z) = I mh(z) + $\overline{(-1)^m (I)^m g(z)}$

(3) Putting μ = 1; in(1.6), the class $\overline{SI^m (1,1,\gamma)}$ reduces to the class

$\overline{SI^m}(l,\gamma)$ ={ $f_{\in S_H} Re \frac{I_l^m h(z)+(-1)^m \overline{(I)^m g(z)}}{z}$} < γ;; 1< γ ≤2; m ∈ $Z = \{0, \pm 1 \pm 2 \pm 3 \pm \cdots$ ... ... z ∈ U}

where I m l is the modified Cho-Kim operator [3] (also see [4]), defined as follows: $I_l^m h(z) + (-1)^m \overline{(I)^m g(z)}$

(4) Putting l = 0; in(1.6), the class $\overline{SI^m (\mu,\gamma)}$ reduces to the class

$\overline{SI^m} (\mu,\gamma)$: ={ $f_{\in S_H} Re \frac{D^m h(z)+\overline{(-1)^m (D)^m g(z)}}{z}$} < γ; 1< γ ≤2; m ∈ $\mathbb{N}_0$ z ∈ U}

where D_μ^m is the modified Al-Oboudi operator (see [1]), deÖned as follows:

$D_\mu^m f(z) = D_\mu^m h(z) + (1)m \overline{D_\mu^m g(z)}$: .

2. COEFFICIENT ESTIMATES

Unless otherwise mentioned, we shall assume in the reminder of this paper that, the parameters 1< γ ≤2; m ∈ $\mathbb{N}_0$ and l ≥ 0:

Theorem 1:

Let f = h + $\bar{g}$ be so that h (z) and g (z) given by (1.2). Furthermore,

let $\sum_{n=2}^{\infty} [\frac{\alpha+\mu(n-1)}{\alpha}]^m |a_n| + \sum_{n=2}^{\infty} [\frac{\alpha+\mu(n-1)}{\alpha}]^m |b_n| \le \gamma - 1$(2.1)

Then f (z) is sense-preserving, harmonic univalent in U and f (z) ∈ SIm ((μ, l:γ). Proof.

If z1 ≠z2; then

$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \ge = 1-\left|\frac{\sum_{n=1}^{\infty} b_n(z_1^n - z_2^n)}{(z_1^n - z_2^n)+\sum_{n=1}^{\infty} b_n(z_1^n - z_2^n)}\right|$

$> 1-\left|\frac{\sum_{n=1}^{\infty} n|b_n|}{1-\sum_{n=2}^{\infty} n|b_n|}\right|$

$\ge 1-\left|\frac{\frac{[\frac{\alpha+\mu(n-1)}{\alpha}]^1}{\gamma-1}|b_n|}{\frac{[\frac{\alpha+\mu(n-1)}{\alpha}]^1}{\gamma-1}|a_n|}\right|$

which proves univalence. Note that f (z) is sense-preserving in U. This is because

$|h'(z)| \ge 1 - \sum_{n=2}^{\infty} n|a_n|\,|z^{n-1}|$

$> 1 - \sum_{n=2}^{\infty} n|a_n|$

$\ge 1- \sum_{n=2}^{\infty} \frac{[\frac{\alpha+\mu(n-1)}{\alpha}]^1}{\gamma-1} |b_n| \ge \sum_{n=2}^{\infty} n|a_n|$

$$\geq \sum_{n=1}^{\infty} \frac{[\frac{\alpha+\mu(n-1)}{\alpha}]^{1}}{\gamma-1} |b_n| \geq \sum_{n=1}^{\infty} n|b_n|$$

$$> \sum_{n=1}^{\infty} n|a_n|\,|z^{n-1}|$$

$$\geq |g'(z)|$$

Now we will show that f (z) ∈ SIm(μ; l; γ). We only need to show that if (2.1) holds then the condition (1.6) is satisfied.

Using the fact that Re {ω} < if and only if $|w-1| < |w-(2\gamma-1)|$ it is suffices to show that

$$\left| \frac{\frac{I^m (\mu,l)h(z)+\overline{(-1)^m (\mu,l)g(z)}}{z} - 1}{\frac{I^m (\mu,l)h(z)+\overline{(-1)^m (\mu,l)g(z)}}{z} - (2\gamma-1)} \right|$$

< 1:

We have

$$\left| \frac{\frac{I^m (\mu,l)h(z)+\overline{(-1)^m (\mu,l)g(z)}}{z} - 1}{\frac{I^m (\mu,l)h(z)+\overline{(-1)^m (\mu,l)g(z)}}{z} - (2\gamma-1)} \right|$$

$$= \left| \frac{\frac{\sum_{n=2}^{\infty} \frac{\alpha+\mu(n-1)}{\alpha}]^m a_n z^{n-1} + \sum_{n=1}^{\infty} \frac{\alpha+\mu(n-1)}{\alpha}]^m \overline{b_n z^{n-1}}}{z}}{(2\gamma-1)+\sum_{n=2}^{\infty} \frac{\alpha+\mu(n-1)}{\alpha}]^m a_n z^{n-1} + \sum_{n=1}^{\infty} \frac{\alpha+\mu(n-1)}{\alpha}]^m \overline{b_n z^{n-1}}} \right|$$

$$\leq \frac{\frac{\sum_{n=2}^{\infty} \frac{\alpha+\mu(n-1)}{\alpha}]^m |a_n||z^{n-1}| + \sum_{n=1}^{\infty} \frac{\alpha+\mu(n-1)}{\alpha}]^m \overline{|b_n||z^{n-1}|}}{z}}{(2\gamma-1)+\sum_{n=2}^{\infty} \frac{\alpha+\mu(n-1)}{\alpha}]^m |a_n||z^{n-1}| + \sum_{n=1}^{\infty} \frac{\alpha+\mu(n-1)}{\alpha}]^m \overline{|b_n||z^{n-1}|}} < \frac{\frac{\sum_{n=2}^{\infty} \frac{\alpha+\mu(n-1)}{\alpha}]^m a_n z^{n-1} + \sum_{n=1}^{\infty} \frac{\alpha+\mu(n-1)}{\alpha}]^m \overline{b_n z^{n-1}}}{z}}{(2\gamma-1)+\sum_{n=2}^{\infty} \frac{\alpha+\mu(n-1)}{\alpha}]^m a_n z^{n-1} + \sum_{n=1}^{\infty} \frac{\alpha+\mu(n-1)}{\alpha}]^m \overline{b_n z^{n-1}}} \leq 1$$

which is bounded above by 1 by using (2.1). This completes the proof of Theorem 1.

Theorem 2. A function f(z) of the form (1.1) is in the class $\overline{SI^m}$(μ; l; γ). if and only if

$$\sum_{n=2}^{\infty} [\frac{\alpha+\mu(n-1)}{\alpha}]^m |a_n| + \sum_{n=1}^{\infty} [\frac{\alpha+\mu(n-1)}{\alpha}]^m |b_n| \leq \gamma - 1:$$

Proof. Since $\overline{SI^m}$(μ; l; γ)⊂SIm(μ; l; γ), we only need to prove the "only if" part of this theorem. To this end , for functions f(z) of the formSubclass of Harmonic Univalent Functions Defined by Modified Cataís Operator

$$\text{Re}\left\{\frac{I^m h(z)+\overline{(-1)^m (l)^m g(z)}}{z}\right\} < \gamma$$

is equivalent to

$$\text{Re}\left\{\sum_{n=2}^{\infty} \frac{\alpha+\mu(n-1)}{\alpha}]^m a_n z^{n-1} + \sum_{n=1}^{\infty} \frac{\alpha+\mu(n-1)}{\alpha}]^m b_n \overline{z^{n-1}}\right\}$$

$$\leq \sum_{n=2}^{\infty} \frac{\alpha+\mu(n-1)}{\alpha}]^m a_n z^{n-1} + \sum_{n=1}^{\infty} \frac{\alpha+\mu(n-1)}{\alpha}]^m b_n \overline{z^{n-1}}$$

$$< 1 + \frac{\sum_{n=2}^{\infty}\frac{\alpha+\mu(n-1)}{\alpha}]^m|a_n||z^{n-1}| + \sum_{n=1}^{\infty}\frac{\alpha+\mu(n-1)}{\alpha}]^m\overline{|b_n||z^{n-1}|}}{z} < \gamma .$$

Letting $z \to 1^-$along the real axis, we obtain the inequality (2.1). This completes the proof of Theorem Theorem 2. {1].

3. DISTORTION THEOREM

Theorem 3:

Let the function f(z) defined by (1.1) belong to the class SIm(μ; l; γ). Then for $|z| = r < 1$;we have

$(1 - |b_1|)\, r - \left[\frac{\alpha}{\alpha+\mu}\right]^m (\gamma - 1 - |b_1|)\, r2 \leq |f(z)|$

$\leq (1 + |b_1|)\, r + [\frac{\alpha}{\alpha+\mu}]^m (\gamma - 1 - |b_1|)\, r2 \leq |f(z)|($ (3.1)

for $|b_1| \leq \gamma - 1$:

The results are sharp with equality for the functions f(z) defined by

$f(z) = z + b1\bar{z} - \left[\frac{\alpha}{\alpha+\mu}\right]^m (\gamma - 1 - |b_1|)$ z-2 (3.2) and $f(z) = z - b1\bar{z} - \left[\frac{\alpha}{\alpha+\mu}\right]^m (\gamma - 1 - |b_1|)$ z-2 (3.3)

Proof. We only prove the right-hand inequality. The proof for the left-hand inequality is similar and will be omitted.

Let $f(z) \in \overline{SI^m}$ (μ; l; γ). Taking the absolute value of we have

$|f(z)| \leq (1 - |b_1|)\, r + \sum_{n=2}^{\infty}(|a_n| + |b_n|)$ rn

$\leq (1 - |b_1|)\, r + \sum_{n=2}^{\infty}(|a_n| + |b_n|)$ r2

$=(1 + |b_1|)\, r + \frac{(\gamma-1)(\alpha)^m}{(\alpha+\mu)^m}\sum_{n=2}^{\infty}\left(\frac{(\alpha+\mu)^m}{(\gamma-1)(\alpha)^m}|a_n| + \frac{(\alpha+\mu)^m}{(\gamma-1)(\alpha)^m}|b_n|\right) r^2$

$\leq (1 + |b_1|)\, r + \frac{(\gamma-1)(\alpha)^m}{(\alpha+\mu)^m}\sum_{n=2}^{\infty}\left(\frac{(\alpha+\mu(n-1))^m}{(\gamma-1)(\alpha)^m}|a_n| + \frac{(\alpha+\mu(n-1))^m}{(\gamma-1)(\alpha)^m}|b_n|\right) r^2$

$\leq (1 + |b_1|)\, r + \frac{(\gamma-1)(\alpha)^m}{(\alpha+\mu)^m}(1 + \frac{|b_1|}{\gamma-1`})r^2$

$\leq (1 + |b_1|)\, r + \frac{(\alpha)^m}{(\alpha+\mu)^m}(\gamma - 1 + |b_1|)r^2$

Similarly we can prove $|f(z)| \geq (\gamma - 1 + |b_1|)r - \frac{(\alpha)^m}{(\alpha+\mu)^m}(\gamma - 1 + |b_1|)r^2$ The functions f(z) given by (3.2) and (3.3), respectively, for jb1j 1 show that the bounds given in Theorem 3 are sharp.Extreme points

Theorem 4:

Let f (z) be given by (1.1). Then f(z) ∈SIm(μ; l; γ). if and only if

$$f(z) = \sum_{n=1}^{\infty}(\mu_n h_n(z) + \eta_n g_n(z)) \text{ rn } (4.1)$$

where h1(z) = z;

$$hn(z) = z + \left[\frac{\alpha}{\alpha+\mu(n-1)}\right]^m (\gamma-1)zn \quad (n\geq 2: m \in \mathbb{N}_0 (4.2)$$

and

$$gn(z) = z + (-1)^m \left[\frac{\alpha}{\alpha+\mu(n-1)}\right]^m (\gamma-1)\overline{z^n} \quad (n\geq 2: m \in \mathbb{N}_0((4.3)$$

$\mu_n \geq 0, \eta_n \geq 0$,; $\sum_{n=1}^{\infty}(\mu_n + \eta_n)$= 1: In particular, the extreme points of theclass $\overline{SIm}(\mu; l; \gamma)$are $\{h_n\}$and $\{g_n\}$ respectively.

5. CONVOLUTION AND CONVEX COMBINATION

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form:

f (z) = z + $\sum_{n=2}^{\infty}|a_n|z^n + \sum_{n=1}^{\infty}|b_n|\overline{z^n}$ (5.1) and

F (z) = z + $\sum_{n=2}^{\infty}|A_n|z^n + \sum_{n=1}^{\infty}|B_n|\overline{z^n}$ (5.2)

the convolution of f and F is given by

(f* F) (z) = f (z) *F (z) = z ++ $\sum_{n=2}^{\infty}|a_nA_n|z^n + \sum_{n=1}^{\infty}|a_nB_n|\overline{z^n}$: (5.3) Using this definition, the next theorem shows that the class $B_n$is closed under convolution. Theorem 5:

For 1 < $\gamma \leq \lambda \leq$ 2; let $\overline{SI^m}(\mu; l; \gamma)$where f (z) is given by (5.1) and

F$\in$ $\overline{SI^m}(\mu; l; \gamma)$where F (z) is given by (5.2). Then $\overline{SI^m}(\mu; l; \gamma) \subset \overline{SI^m}(\mu; l; \gamma)$

Theorem 6:

The class $\overline{SI^m}(\mu; l; \gamma)$ is closed under convex combination

6. A FAMILY OF INTEGRAL OPERATORS

Theorem 7:

Let the function f (z) defined by (1.1) be in the class $\overline{SI^m}(\mu; l; \gamma)$ and let c be a real number such that c > -1: Then the function F (z) defined by

F (z) = $\frac{c+1}{z^c}\int_0^z t^{c-1}$ h (t) dt + $\frac{c+1}{z^c}\int_0^z t^{c-1}$ h (t) dt (c > 1) (6.1) also belongs to the class $\overline{SI^m}(\mu; l; \gamma)$

Remark 2:

Specializing the parameters l; and m, in the above results, we obtain the corresponding results for the corresponding classes de Öned in the introduction.

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